(As in the lecture) 9/

i) $\delta_V = \delta + i[V,\cdot]$, $\mathcal{D}(\delta_V) = \mathcal{D}(\delta)$

$$\tau_{-s}(\tau_s^V(A)) = \frac{d}{ds}\, e^{-s\delta}\left(e^{s\delta_V}(A)\right)$$

$$= -\delta\, e^{-s\delta}\left(e^{s\delta_V}(A)\right) + e^{-s\delta}\left(\delta + i[V,\cdot]\right)e^{s\delta_V}(A)$$

$$= e^{-s\delta}\, i[V,\cdot]\, e^{s\delta_V}(A) = \tau_{-s}\left(i[V,\tau_s^V(A)]\right)$$

$$\tau_t^V(A) - \tau_t(A) = \tau_{t-s}\left(\tau_s^V(A)\right)\Big|_{s=0}^{s=t}$$

$$= \int \tau_{t-s}\left(i[V,\tau_s^V(A)]\right) ds$$

$$\tau_t^V(A) = \tau_t(A) + \int_0^t i[\tau_{t-s}(V), \tau_s^V(A)]\, ds$$

(Duhamel)

ting this formula N times yields

$$(A) = \tau_t(A) + \sum_{k=1}^{N} i^k \int_{0\le t_1\le\dots\le t_k\le t} [\tau_{t_1}(V),[\dots,[\tau_{t_k}(V),\tau_t(A)]]]\, dt_1\dots dt_k$$

$$\underbrace{i^{N+1}\int_{0\le t_1\le\dots\le t_N\le s\le t} [\tau_{t_1}(V),[\dots,[\tau_{t_N}(V),\tau_s^V(A)]]]\, dt_1\dots dt_N\, ds}_{=:\, R_N(A,t)}$$

Pf

$$\|R_N(A,t)\| \le 2^{N+1}\Big(\sup_{t\in[0,T]} \|\tau_t(V)\|\Big)^{N+1} \|A\| \frac{t^{N+1}}{(N+1)!}$$

This is seen by estimating each commutator $\|[T,S]\| \le 2\|T\|\|S\|$ and observing that

$$\int 1\!\!1\{0 \le t_1 \le \dots \le t_N \le s \le t\}\, dt_1 \dots dt_N ds$$

$$= \frac{t^{N+1}}{(N+1)!}$$

$$\Rightarrow \sup_{t\in[0,T]} \|R_N(A,t)\| \le \frac{C_{T,V}^{N+1}}{(N+1)!} \|A\|$$

for some const. $C_{T,V} > 0$.

$$\Rightarrow \sup_{t\in[0,T]} \|R_N(A,t)\| \xrightarrow{N\to\infty} 0.$$

ii) $\tau_V^t(A)\Gamma_V(t) = \Gamma_V(t)\tau_0^t(A) \quad (\forall t \forall A)$

$$\overset{d/dt}{\Rightarrow} \underbrace{\tau_V^t(\delta_V(A))\Gamma_V(t)}_{} + \tau_V^t(A)\dot\Gamma_V(t) = \dot\Gamma_V(t)\tau_0^t(A) + \Gamma_V(t)\tau_0^t(\delta_0(A))$$

$$= \Gamma_V(t)\tau_0^t(\delta_V(A))$$

$$\Rightarrow \Gamma_V(t)\tau_0^t(i[V,A]) + \tau_V^t(A)\dot\Gamma_V(t) = \dot\Gamma_V(t)\tau_0^t(A)$$

$$\Rightarrow \tau_0^t(i[V,A]) + \tau_0^t(A)\Gamma_V(t)^*\dot\Gamma_V(t) - \Gamma_V(t)^*\dot\Gamma_V(t)\tau_0^t(A) = 0$$

$$\Rightarrow \left[\tau_0^t(A), \Gamma_V(t)^*\dot\Gamma_V(t) - i\tau_0^t(V)\right] = 0$$

$$\Rightarrow \quad \dot{\Gamma}_V(t) = i\,\Gamma_V(t)\,\tau_0^t(V) \quad , \quad \Gamma_V(0) = \mathbb{1} .$$

We repeadly used that $\Gamma_V(t)$ is unitary. In the last step we used that the identity holds for all $A \in \mathcal{A}$.

iii) all three functions (of t) satisfy the same initial value problem. Hence, by uniqueness, all have to be identical.